Path Independence, Potential Functions, and Conservative Fields

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Overview

In gravitational and electric fields, the amount of work it takes to move a mass or a charge from one point to another depends only on the object's initial and final positions and not on the path taken in between.

We discuss in the lecture the notion of path independence of work integrals and the properties of fields in which work integrals are path independent.

Work integrals are often easier to evaluate if they are path independent.

Path Independence

If A and B are two points in an opan region D in space, the work $\int \mathbf{F}.d\mathbf{r}$ done in moving a particle from A to B by a field F defined on D usually depends on the path taken.

For some special fields, however, the integral's value is the same for all paths from A to B.

Definition 1.

Let F *be a field defined on an open region D in space, and suppose that for any two points A and B in D the work*

$$\int_A^B \mathbf{F}.\mathrm{d}\mathbf{r}$$

done in moving from A to B is the same over all paths from A to B. Then the integral F.dr *is* path independent in D *and the field* F *is* conservative on D.

Path Independence

The word *conservative* from physics, where it refers to fields in which the principle of conservation of energy holds (it does, in conservative fields).

Under differntiability conditions normally met in practice, a field F is conservative if and only if it is the gradient field of a scalar function f ; that is, if and only if $\mathrm{F} = \nabla \mathrm{f}$ for some f.

The function f then has a special name.

Definition 2.

If F *is a field defined on* D *and* $\mathrm{F} = \nabla f$ *for some scalar function* f *on* D, *then* f *is called a* potential function for F.

Path Independence

An electric potential is a scalar function whose gradient field is an electric field. A gravitational potential is a scalar function whose gradient field is a gravitational field, and so on.

We shall see some remarkable properties of conservative fields. For example, saying that F is conservative on D is equivalent to saying that the integral of F around every closed path in D is zero.

We shall see that once we have found a potential function f of a field F, we can evaluate all the work integrals in the domain of F over any path between A and B by

$$\int_A^B \mathrm{F}.d\mathrm{r} = \int_A^B \nabla f.d\mathrm{r} = f(B) - f(A). \tag{1}$$

Path Independence

The above equation is the vector calculus analogue of the Fundamental Theorm of Calculus formula (if we think of ∇f for functions of several variables as being something like the derivative f' for functions of a single variables)

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

We discuss the following certain conditions on the curves, fields, and domains to be satisfied for **the equation (1) to be valid.**

Conditions on Curves :

We assume that all curves are **piecewise smooth**, that is, made up of finitely many smooth pieces connected end to end.

Conditions on Fields :

We also assume that the components of **F** have continuous first partial derivatives.

When $\mathbf{F} = \nabla f$, the continuity requirement guarantees that the mixed second derivatives of the potential function f are equal. That is,

$$\frac{\partial^2 M}{\partial x \partial y} = \frac{\partial^2 M}{\partial y \partial x} \quad \text{and so on.}$$

Conditions on Domains

Conditions on Domains :

We assume D to be an **open region** in space. This means that every point in D is the center of an open ball that lies entirely in D.

We assume D to be **connected**, which in an open region means that every point can be connected to every other point by a smooth curve that lies in the region.

Finally, we assume that D is **simply connected**, which means every loop in D can be contracted to a point in D without ever leaving D.

If D consisted of space with a line segment removed, for example, D would not be simply connected. There would be no way to contract a loop around the line segment to a point without leaving D.

Conditions on Domains

Connectivity and simple connectivity are not the same, and neither implies the other.

Think of connected regions as being in "one piece" and simply connected regions as not having any "holes that catch loops."

All of space itself is both connected and simply connected.

Some of the results can faild to hold if applied to domains where these conditions do not hold. For example, the component test for conservative fields is not valid on domains that are not simply connected.

Line Integrals in Conservative Fields

The following result provides a convenient way to evaluate a line integral in a conservative field. The result establishes that the value of the integral depends only on the endpoints and not on the specific path joining them.

Theorem 3 (The Fundamental Theorem of Line Integrals).

Let $\mathsf{F} = M\mathsf{i} + N\mathsf{j} + P\mathsf{k}$ *be a vector field whose components are continuous throughout an open connected region* D *in space. Then there exists a differentiable function* f *such that*

$$\mathsf{F} = \nabla f = \frac{\partial f}{\partial x}\mathsf{i} + \frac{\partial f}{\partial y}\mathsf{j} + \frac{\partial f}{\partial z}\mathsf{k}$$

if and only if for all points A *and* B *in* D *the value of*

$$\int_A^B \mathsf{F}.\mathsf{dr}$$

is independent of the path joining A *to* B *in* D*.*

Line Integrals in Conservative Fields

If the integral is independent of the path from A to B, its value is

$$\int_A^B \mathsf{F}.d\mathsf{r} = f(B) - f(A).$$

Theorem 4 (Closed-Loop Property of Conservative Fields).

The following statements are equivalent.

1. $\oint \mathsf{F}.d\mathsf{r} = 0$ *around every closed loop in* D.
2. *The field* F *is conservative on* D.

Line Integrals in Conservative Fields

We summarize the results of the above two theorems as follows: The following statements are equivalent.

1. $\mathbf{F} = \nabla f$ on D, for some scalar function f on D.
2. $\mathbf{F}$ is conservative on D.
3. $\oint_C \mathbf{F}.d\mathbf{r} = 0$, for every closed path C in D.

A test for being conservative

Theorem 5.

Let $\mathrm{F} = M(x, y, z)\mathrm{i} + N(x, y, z)\mathrm{j} + P(x, y, z)\mathrm{k}$ *be a field whose component functions have continuous first partial derivatives and domain of* F *is connected and simply connected. Then,* F *is conservative if and only if*

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

If the component functions of **F** satisfy the above three equations, then the given field **F** is conservative and vice versa.

The test is called "Component Test for Conservative Fields." The component test for conservative fields is not valid on domains that are not simply connected.

If $\mathbf{F}$ is conservative, how do we find a potential function f (so that $\mathbf{F} = \nabla f$)?

Once we know that $\mathbf{F}$ is conservative, we usually want to find a potential function for $\mathbf{F}$.

This requires solving the equation $\nabla f = \mathbf{F}$ or

$$\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$$

for f.

We can find a potential function by integrating the three equations

$$\frac{\partial f}{\partial x} = M, \quad \frac{\partial f}{\partial y} = N, \quad \frac{\partial f}{\partial z} = P.$$

Exact Differential Forms

Sometimes it is convenient to express work and circulation integrals in the "differential" form

$$\int_A^B M\ dx + N\ dy + P\ dz.$$

Such integrals are relatively easy to evaluate $M\ dx + N\ dy + P\ dz$ is the total differential of a function f.

Definition 6.

Any expression $M(x,y,z)dx + N(x,y,z)dy + P(x,y,z)dz$ *is a* differential form. *A differential form is* exact *on a domain* D *in space if*

$$M\ dx + N\ dy + P\ dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\frac{\partial f}{\partial z}dz = df$$

for some scalar function f *throughout* D.

Component Test for Exactness of $M\ dx + N\ dy + P\ dz$

If $M\ dx + N\ dy + P\ dz = df$ on D, then $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is the **gradient field** of f on D.

Conversely, if $\mathbf{F} = \nabla f$, then the form $M\ dx + N\ dy + P\ dz$ is exact.

The test for the (differential) form's being exact is therefore same as the test for **F**'s being conservative.

Equivalent : Conservative and Exact

Let $\mathbf{F} = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ be a field whose component functions have continuous first partial derivatives and domain of $\mathbf{F}$ is connected and simply connected.

The following statements are equivalent.

1. The field F is conservative.
2. $M\ dx + N\ dy + P\ dz$ is exact.
3. $\dfrac{\partial P}{\partial y} = \dfrac{\partial N}{\partial z}, \quad \dfrac{\partial M}{\partial z} = \dfrac{\partial P}{\partial x}, \quad \dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}.$

Testing for Conservative Fields

Exercise 7.

Which fields in the following exercises are conservative, and which are not?

1. $F = yi + (x+z)\,j - yk$
2. $F = (e^x \cos y)\, i - (e^x \sin y)\, j + zk$

Solution for Exercise 7

1. $\frac{\partial P}{\partial y} = -1 \neq 1 = \frac{\partial N}{\partial z} \Rightarrow$ Not Conservative
2. $\frac{\partial p}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -e^x \sin\ y = \frac{\partial M}{\partial y} \Rightarrow$ Conservative

Exercise 8.

In the following exercises, find a potential function f for the field F.

1. $F = 2xi + 3yj + 4zk$
2. $F = \left(\ln x + \sec^2(x+y)\right) i + \left(\sec^2(x+y) + \frac{y}{y^2+z^2}\right) j + \frac{z}{y^2+z^2} k$
3. $F = \frac{y}{1+x^2y^2} i + \left(\frac{x}{1+x^2y^2} + \frac{z}{\sqrt{1-y^2z^2}}\right) j + \left(\frac{y}{\sqrt{1-y^2z^2}} + \frac{1}{z}\right) k.$

Solution for Exercise 8

1. $\frac{\partial f}{\partial x} = 2x \Rightarrow f(x,y,z) = x^2 + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = 3y \Rightarrow g(y,z) = \frac{3y^2}{2} + h(z) \Rightarrow f(x,y,z) = x^2 + \frac{3y^2}{2} + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = 4z \Rightarrow h(z) = 2z^2 + C \Rightarrow f(x,y,z) = x^2 + \frac{3y^2}{2} + 2z^2 + C$
2. $\frac{\partial f}{\partial z} = \frac{z}{y^2+z^2} \Rightarrow f(x,y,z) = \frac{1}{2}\ln(y^2+z^2) + g(x,y) \Rightarrow \frac{\partial f}{\partial x} = \frac{\partial g}{\partial x} = \ln\ x + \sec^2(x+y) \Rightarrow g(x,y) = (x\ \ln\ x - x) + \tan(x+y) + h(y) \Rightarrow f(x,y,z) = \frac{1}{2}\ln(y^2+z^2) + (x\ \ln\ x - x) + \tan(x+y) + h(y) \Rightarrow \frac{\partial f}{\partial y} = \frac{y}{y^2+z^2} + \sec^2(x+y) + h'(y) = \sec^2(x+y) + \frac{y}{y^2+z^2} \Rightarrow h'(y) = 0 \Rightarrow h(y) = C \Rightarrow f(x,y,z) = \frac{1}{2}\ln(y^2+z^2) + (x\ \ln\ x - x) + \tan(x+y) + C$
3. $\frac{\partial f}{\partial x} = \frac{y}{1+x^2y^2} \Rightarrow f(x,y,z) = \tan^{-1}(xy) + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{x}{1+x^2y^2} + \frac{\partial g}{\partial y} = \frac{x}{1+x^2y^2} + \frac{z}{\sqrt{1-y^2z^2}} \Rightarrow \frac{\partial g}{\partial y} = \frac{z}{\sqrt{1-y^2z^2}} \Rightarrow g(y,z) = \sin^{-1}(yz) + h(z) \Rightarrow f(x,y,z) = \tan^{-1}(xy) + \sin^{-1}(yz) + h(z) \Rightarrow \frac{\partial f}{\partial z} = \frac{y}{\sqrt{1-y^2z^2}} + h'(z) = \frac{y}{\sqrt{1-y^2z^2}} + \frac{1}{z} \Rightarrow h'(z) = \frac{1}{z} \Rightarrow h(z) = \ln|z| + C \Rightarrow f(x,y,z) = \tan^{-1}(xy) + \sin^{-1}(yz) + \ln|z| + C$

Exercise 9.

In the following exercises, show that the differential forms in the integrals are exact. Then evaluate the integrals.

1. $\displaystyle\int_{(0,0,0)}^{(2,3,-6)} 2x\ dx + 2y\ dy + 2z\ dz$
2. $\displaystyle\int_{(0,0,0)}^{(1,2,3)} 2xy\ dx + \left(x^2 - z^2\right) dy - 2yz\ dz$
3. $\displaystyle\int_{(1,0,0)}^{(0,1,1)} \sin y \cos x\ dx + \cos y \sin x\ dy + dz$

Solution for Exercise 9

1. Let $\mathbf{F}(x,y,z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow \frac{\partial p}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial p}{\partial x}, \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y} \Rightarrow$ $M\ dx + N\ dy + P\ dz$is exact; $\frac{\partial f}{\partial x} = 2x \Rightarrow f(x,y,z) = x^2 + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = 2y \Rightarrow$ $g(y,z) = y^2 + h(z) \Rightarrow f(x,y,z) = x^2 + y^2 = h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = 2z \Rightarrow h(z) =$ $z^2 + C \Rightarrow f(x,y,z) = x^2 + y^2 + z^2 + C \Rightarrow \int_{(0,0,0)}^{(2,3,-6)} 2x\ dx + 2y\ dy + 2z\ dz =$ $f(2,3,-6) - f(0,0,0) = 2^2 + 3^2 + (-6)^2 = 49$
2. Let $\mathbf{F}(x,y,z) = 2xy\mathbf{i} + (x^2 - y^2)\mathbf{j} - 2yz\mathbf{k} \Rightarrow \frac{\partial p}{\partial y} = -2z = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 2x = \frac{\partial M}{\partial y} \Rightarrow$ $M\ dx + N\ dy + P\ dz$is exact; $\frac{\partial f}{\partial x} = 2xy \Rightarrow f(x,y,z) = x^2y + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = x^2 + \frac{\partial g}{\partial y} =$ $x^2 - z^2 \Rightarrow \frac{\partial g}{\partial y} = -z^2 \Rightarrow g(y,z) = -yz^2 + h(z) \Rightarrow f(x,y,z) = x^2y - yz^2 + h(z) \Rightarrow \frac{\partial f}{\partial z} =$ $-2yz + h'(z) = -2yz \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x,y,z) = x^2y - yz^2 + C \Rightarrow$ $\int_{(0,0,)}^{(1,2,3)} 2xy\ dx + (x^2 - z^2)dy - 2yz\ dz = f(1,2,3) - f(0,0,0) = 2 - 2(3)^2 = -16$
3. Let $\mathbf{F}(x,y,z) = (\sin y\ \cos\ x)\mathbf{i} + (\cos\ y\ \sin\ x)\mathbf{j} + \mathbf{k} \Rightarrow \frac{\partial p}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 =$ $\frac{\partial p}{\partial x}, \frac{\partial N}{\partial x} = \cos\ y\ \cos\ x = \frac{\partial M}{\partial y} \Rightarrow M\ dx + N\ dy + P\ dz$is text; $\frac{\partial f}{\partial x} = \sin\ y\ \cos\ x \Rightarrow$ $f(x,y,z) = \sin\ y\ \sin\ x + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \cos\ y\ \sin\ x + \frac{\partial g}{\partial y} = \cos\ y\ \sin\ x \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow$ $g(y,z) = h(z) \Rightarrow f(x,y,z) = \sin\ y\ \sin\ x + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = 1 \Rightarrow h(z) = z + C \Rightarrow$ $f(x,y,z) = \sin\ y\ \sin\ x + z + C \Rightarrow \int_{(1,0,0)}^{(0,1,1)} \sin\ y\ \cos\ x\ dx + \cos\ y\ \sin\ x\ dy + dz =$ $f(0,1,1) - f(1,0,0) = (0+1) - (0+0) = 1$

Finding Potential Functions to Evaluate Line Integrals

Exercise 10.

Although they are not defined on all of space R^3, the fields associated with the following exercises are simply connected and the Component Test can be used to show they are conservative. Find a potential function for each field and evaluate the integrals.

1. $\displaystyle\int_{(0,2,1)}^{(1,\pi/2,2)} 2\cos y \; dx + \left(\frac{1}{y} - 2x\sin y\right) dy + \frac{1}{z} dz$
2. $\displaystyle\int_{(1,2,1)}^{(2,1,1)} (2x\ln y - yz)\, dx + \left(\frac{x^2}{y} - xz\right) dy - xy \; dz$
3. $\displaystyle\int_{(-1,-1,-1)}^{(2,2,2)} \frac{2x \; dx + 2y \; dy + 2z \; dz}{x^2 + y^2 + z^2}$

Solution for Exercise 10

1. Let $\mathbf{F}(x,y,z) = (2\cos y)\mathbf{i} + \left(\frac{1}{y} - 2x\sin y\right)\mathbf{j} + \left(\frac{1}{z}\right)\mathbf{k} \Rightarrow \frac{\partial p}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -2\sin y = \frac{\partial M}{\partial y} \Rightarrow M\,dx + N\,dy + P\,dz$ is exact; $\frac{\partial f}{\partial x} = 2\cos y \Rightarrow f(x,y,z) = 2x\cos y + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = -2x\sin y + \frac{\partial g}{\partial y} = \frac{1}{y} - 2x\sin y \Rightarrow \frac{\partial g}{\partial y} = \frac{1}{y} \Rightarrow g(y,z) = \ln|y| + h(z) \Rightarrow f(x,y,z) = 2x\cos y + \ln|y| + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = \frac{1}{z} = h(z) = \ln|z| + C \Rightarrow f(x,y,z) = 2x\cos y + \ln|y| + \ln|z| + C$
   $\Rightarrow \int_{(0,2,1)}^{(1,\pi/2,2)} 2\cos y\,dx + \left(\frac{1}{y} - 2x\sin y\right)dy + \frac{1}{z}dz = f(1,\frac{\pi}{2},2) - f(0,2,1)$
   $= (2.0 + \ln\frac{\pi}{2} + \ln 2) - (0.\cos 2 + \ln 2 + \ln 1) = \ln\frac{\pi}{2}$
2. Let $\mathbf{F}(x,y,z) = (2x\ln y - yz)\mathbf{i} + \left(\frac{x^2}{y} - xz\right)\mathbf{j} - (xy)\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = -x = \frac{\partial N}{\partial z} = \frac{\partial M}{\partial z} = -y = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \frac{2x}{y} - z = \frac{\partial M}{\partial y} \Rightarrow M\,dx + N\,dy + P\,dz$is exact; $\frac{\partial f}{\partial x} = 2x\ln y - yz \Rightarrow f(x,y,z) = x^2\ln y - xyz + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{x^2}{y} - xz + \frac{\partial g}{\partial y} = \frac{x^2}{y} - xz \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z) \Rightarrow f(x,y,z) = x^2\ln y - xyz + h(z) \Rightarrow \frac{\partial f}{\partial z} = -xy + h'(z) = -xy \Rightarrow h'(z) = 0$
   $\Rightarrow h(z) = C \Rightarrow f(x,y,z) = x^2\ln y - xyz + C \Rightarrow \int_{(1,2,1)}^{(2,1,1)} (2x\ln y - yz)dx + \left(\frac{x^2}{y} - xz\right)dy - xy\,dz$
   $= f(2,1,1) - f(1,2,1) = (4\ln 1 - 2 + C) - (\ln 2 - 2 + C) = -\ln 2.$
3. Let $\mathbf{F}(x,y,z) = \frac{2xi+2yj+2zk}{x^2+y^2+z^2}$ (and let $\rho^2 = x^2+y^2+z^2 \Rightarrow \frac{\partial\rho}{\partial x} = \frac{x}{\rho}, \frac{\partial\rho}{\partial y} = \frac{y}{\rho}, \frac{\partial\rho}{\partial z} = \frac{z}{\rho}) \Rightarrow \frac{\partial P}{\partial y} = -\frac{4yz}{\rho^4} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = -\frac{4xz}{\rho^4} = \frac{\partial p}{\partial x}, \frac{\partial N}{\partial x} = -\frac{4xy}{\rho^4} = \frac{\partial M}{\partial y} \Rightarrow M\,dx + N\,dy + P\,dz$is exact; $\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2+z^2} \Rightarrow f(x,y,z) = \ln(x^2+y^2+z^2) + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2+z^2} + \frac{\partial g}{\partial y} = \frac{2y}{x^2+y^2+z^2}$
   $\Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z) \Rightarrow f(x,y,z) = \ln(x^2+y^2+z^2) + h(z) \Rightarrow \frac{\partial f}{\partial z}$
   $= \frac{2z}{x^2+y^2+z^2} + h'(z) = \frac{2z}{x^2+y^2+z^2} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x,y,z) = \ln(x^2+y^2+z^2) + C$
   $\Rightarrow \int_{(-1,-1,-1)}^{(2,2,2)} \frac{2x\,dx+2y\,dy+2z\,dz}{x^2+y^2+z^2} = f(2,2,2) - f(-1,-1,-1) = \ln 12 - \ln 3 = \ln 4$

Exercise 11.

Evaluate the integral

$$\int_{(1,1,1)}^{(2,3,-1)} y\ dx + x\ dy + 4\ dz$$

by finding parametric equations for the line segment form $(1,1,1)$ *to* $(2,3,-1)$ *and evaluating the line integral of* $F = yi + xj + 4k$ *along the segment. Since* F *is conservative, the integral is independent of the path.*

Solution for Exercise 11

$r = (i + j + k) + t(i + 2j - 2k) = (1 + t)i + (1 + 2t)j + (1 - 2t)k, 0 \leq t \leq 1 \Rightarrow dx = dt, dy = 2\ dt, dz = -2\ dt \Rightarrow \int_{(1,1,1)}^{(2,3,-1)} y\ dx + x\ dy + 4\ dz = \int_0^1 (2t + 1)dt + (t + 1)(2\ dt) + 4(-2)dt = \int_0^1 (4t - 5)dt = [2t^2 - 5t]_0^1 = -3$

Exercise 12.

Evaluate

$$\int_c x^2 dx \ + yz \ dy \ + \left(y^2/2\right) dz$$

along the line segment C *joining* $(0, 0, 0)$ *to* $(0, 3, 4)$.

Solution for Exercise 12

$r = t(3j + 4k), 0 \leq t \leq 1 \Rightarrow dx = 0, dy = 3\ dt, dz = 4\ dt$

$\Rightarrow \int_{(0,0,0)}^{(0,3,4)} x^2\ dx + yz\ dy + \left(\frac{y^2}{2}\right) dz = \int_0^1 (12t^2)(3\ dt) + \left(\frac{9t^2}{2}\right)(4\ dt) = \int_0^1 54t^2\ dt = [18t^2]_0^1 = 18$

Exercise 13.

Show that the values of the integrals in the following exercises not depend on the path taken form A to B.

1. $\displaystyle\int_A^B z^2 dx + 2y\ dy + 2xz\ dz$
2. $\displaystyle\int_A^B \frac{x\ dx\ + y\ dy + z\ dz}{\sqrt{x^2 + y^2 + z^2}}$

Solution for Exercise 13

1. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 2z = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y} \Rightarrow M\ dx + N\ dy + P\ dz$is exact $\Rightarrow$ **F**is conservative $\Rightarrow$ path independence
2. $\frac{\partial P}{\partial y} = -\frac{yz}{(\sqrt{x^2+y^2+z^2})^3} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = -\frac{xz}{(\sqrt{x^2+y^2+z^2})^3} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -\frac{xy}{(\sqrt{x^2+y^2+z^3})^3} = \frac{\partial M}{\partial y} \Rightarrow$ $M\ dx + N\ dy + P\ dz$is exact $\Rightarrow$ **F**is conservative $\Rightarrow$ path independence

Exercises

Exercise 14.

In the following exercises, find a potential function for F.

1. $F = \frac{2x}{y} i + \left(\frac{1-x^2}{y^2} \right) j, \quad \{(x, y) : y > 0\}$
2. $F = (e^x \ln y)\, i + \left(\frac{e^x}{y} + \sin z \right) j + (y \cos z)\, k$

Solution for Exercise 14

1. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -\frac{2x}{y^2} = \frac{\partial M}{\partial y} \Rightarrow$ **F**is conservative $\Rightarrow$ there exists an f so that$F = \nabla f$; $\frac{\partial f}{\partial x} = \frac{2x}{y} \Rightarrow f(x,y) = \frac{x^2}{y} + g(y) \Rightarrow \frac{\partial f}{\partial y} = -\frac{x^2}{y^2} + g'(y) = \frac{1-x^2}{y^2} \Rightarrow g'(y) = \frac{1}{y^2} \Rightarrow g(y) = -\frac{1}{y} + C \Rightarrow f(x,y) = \frac{x^2}{y} - \frac{1}{y} + C \Rightarrow \mathbf{F} = \nabla\left(\frac{x^{-}1}{y}\right)$
2. $\frac{\partial P}{\partial y} = \cos\ z = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial y} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \frac{e^x}{y} = \frac{\partial M}{\partial y} \Rightarrow$ **F**is conservative $\Rightarrow$ there exists an f so that$\mathbf{F} = \nabla f$; $\frac{\partial f}{\partial x} = e^x \ln\ y \Rightarrow f(x,y,z) = e^x \ln\ y + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{e^x}{y} + \frac{\partial g}{\partial y} = \frac{e^x}{y} + \sin\ z \Rightarrow \frac{\partial g}{\partial y} = \sin\ z \Rightarrow g(x,z) = y \sin\ z + h(z) \Rightarrow f(x,y,z) = e^x \ln\ y + y\ sin\ z + h(z) \Rightarrow \frac{\partial f}{\partial z} = y\ cos\ z + h'(z) = y\ cos\ z \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x,y,z) = e^x \ln\ y + y\ \sin z + C \Rightarrow \mathbf{F} = \nabla(e^x \ln\ y + y\ sin\ z)$

Work along different paths

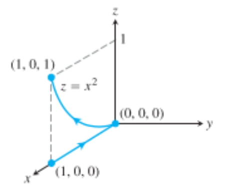
Exercise 15.

Find the work done by $F = (x^2 + y)\, i + (y^2 + x)\, j + ze^z k$ *over the following paths from* $(1, 0, 0)$ *to* $(1, 0, 1)$.

(a) *The line segment* $x = 1, y = 0, 0 \leq z \leq 1$.

(b) *The helix* $r(t) = (\cos t)\, i + (\sin t)\, j + (t/2\pi)\, k, 0 \leq t \leq 2\pi$.

(c) *The x-axis form* $(1, 0, 0)$ *to* $(0, 0, 0)$ *followed by the parabola* $z = x^2, y = 0$ *from* $(0, 0, 0)$ *to* $(1, 0, 1)$.



Solution for Exercise 15

$\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z} \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 1 = . \frac{\partial M}{\partial y} =$ **F**is conservative $\Rightarrow$ there exists an f so that**F** $=$ ∇f; $\frac{\partial f}{\partial x} = x^2 + y \Rightarrow f(x,y,z) = \frac{1}{3}x^3 + xy + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y} = y^2 + x \Rightarrow \frac{\partial g}{\partial y} = y^2 \Rightarrow$ $g(y,z) = \frac{1}{3}y^3 + h(z) \Rightarrow f(x,y,z) = \frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = ze^z \Rightarrow h(z) = ze^z =$ $-e^z + C \Rightarrow f(x,y,z) = \frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z + C \Rightarrow \mathbf{F} = \nabla\left(\frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z\right)$

(a) Work$= \int_A^B \mathbf{F} \cdot \frac{dr}{dt} dt = \int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[\frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z\right]_{(1,0,0)}^{(1,0,1)} =$ $\left(\frac{1}{3} + 0 + 0 + e - e\right) - \left(\frac{1}{3} + 0 + 0 - 1\right) = 1$

(b) Work$= \int_A^B \mathbf{F} \cdot dr = \left[\frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z\right]_{(1,0,0)}^{(1,0,1)} = 1$

(c) Work$= \int_A^B \mathbf{F} \cdot dr = \left[\frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z\right]_{(1,0,0)}^{(1,0,1)} = 1$

Note:Since **F** is conservative, $\int_A^B \mathbf{F} \cdot dr$ is independent of the path from $(1,0,0) to (1,0,1)$

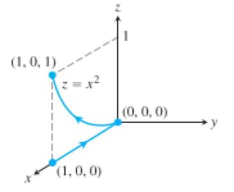
Exercise 16.

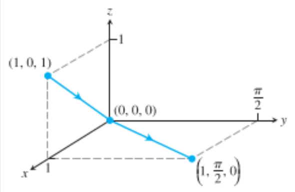
Find the work done by $F = e^{yz}i + (xze^{yz} + z\cos y)j + (xye^{yz} + \sin y)k$ *over the following paths from* $(1, 0, 1)$ *to* $(1, \pi/2, 0)$.

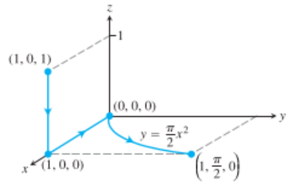
(a) *The line segment* $x = 1, y = \pi t/2, z = 1 - t, 0 \leq t \leq 1$.

(b) *The line segment from* $(1, 0, 1)$ *to the origin followed by the line segment from the origin to* $(1, \pi/2, 0)$.

(c) *The line segment from* $(1, 0, 0)$ *to* $(1, 0, 0)$, *followed by the x-axis from* $(1, 0, 0)$ *to the origin, followed by the parabola* $y = \pi x^2/2, z = 0$ *from there to* $(1, \pi/2, 0)$.







Solution for Exercise 16

$\frac{\partial P}{\partial y} = xe^{yz} + xyze^{yz} + \cos\ y + \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = ye^{yz} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = ze^{yz} = \frac{\partial M}{\partial y} \Rightarrow$ **F**is conservative $\Rightarrow$ there exists an f so that**F** $= \nabla f$; $\frac{\partial f}{\partial x} = e^{yz} \Rightarrow f(x,y,z) = xe^{yz} + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = xze^{yz} + \frac{\partial g}{\partial y} = xze^{yz} + z\ \cos\ y \Rightarrow \frac{\partial g}{\partial y} = z\ \cos\ y \Rightarrow g(y,z) = z\ \sin\ y + h(z) \Rightarrow f(x,y,z) = xe^{yz} + z\ \sin\ y + hz \Rightarrow \frac{\partial f}{\partial z} = xye^{yz} + \sin\ y + h'(z) = xye^{yz} + \sin\ y \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x,y,z) = xe^{yz} + z\ \sin\ y + C \Rightarrow \mathbf{F} = \nabla(xe^{yz} + z\ sin\ y)$

(a) work$= \int_A^B \mathbf{F} \cdot d\mathbf{r} = [xe^{yz} + z\ \sin\ y]_{(1,0,1)}^{(1,\pi/2,0)} = (1+0) - (1+0) = 0$

(b) work$= \int_A^B \mathbf{F} \cdot d\mathbf{r} = [xe^{yz} + z\ \sin\ y]_{(1,0,1)}^{(1,\pi/2,0)} = 0$

(c) work$= \int_A^B \mathbf{F} \cdot d\mathbf{r} = [xe^{yz} + z\ \sin\ y]_{(1,0,1)}^{(1,\pi/2,0)} = 0$

Note:Since **F** is conservative, $\int_A^B \mathbf{F} \cdot dr$ is independent of the path from $(1,0,1)$ to $(1, \frac{\pi}{2}, 0)$.

Evaluating a work integral two ways

Exercise 17.

Let $F = \nabla\left(x^3y^2\right)$ *and let* C *be the path in the xy-plane from* $(-1,1)$ *to* $(1,1)$ *that consists of the line segment from* $(-1,1)$ *to* $(0,0)$ *followed by the line segment form* $(0,0)$ *to* $(1,1)$ *. Evaluate* $\int_c F \cdot dr$ *in two ways.*

(a) *Find parametrizations for the segments that make up* C *and evaluate integral.*

(b) *Use* $f(x,y) = x^3y^2$ *as a potential function for* F.

Solution for Exercise 17

(a) $\mathbf{F} = \nabla(x^3y^2) \Rightarrow \mathbf{F} = 3x^2 + y^2\mathbf{i} + 2x^3y\mathbf{j}$; let C_1 be the path from$(-1,1)to(0,0) \Rightarrow x = t-1$ and $y = -t+1, 0 \le t \le 1 \Rightarrow \mathbf{F} + 3(t-1)^2(-t+1^2)\mathbf{i} + 2(t-1)^3(-t+1)\mathbf{j} = 3(t-1)^4\mathbf{i} - 2(t-1^4)\mathbf{j}$ and $r_1 = (t-1)\mathbf{i} + (-t+1)\mathbf{j} \Rightarrow dr_1 = dt\ \mathbf{i} - dt\ \mathbf{j} \Rightarrow \int_{c_1} \mathbf{F} \cdot dr_1 - \int_0^1 [3(t-1)^4 + 2(t-1)^4]dt = \int_0^1 5(t-1)^4 dt = [(t-1)^5]_0^1 = 1$; let C_2 be the pathe from$(0,0)to(1,1) \Rightarrow x = t$ and $y = t, 0 \le t \le 1 \Rightarrow \mathbf{F} = 3t^4\mathbf{i} + 2t^4\mathbf{j}$ and $\mathbf{r}_2 = t\mathbf{i} + t\mathbf{j} \Rightarrow d\mathbf{r}_2 = dt\ \mathbf{i} + dt\ \mathbf{j} \Rightarrow \int_{c_2} \mathbf{F} \cdot dr_2 = \int_0^1 (3t^2 + 2t^4)dt = \int_0^1 5t^4 dt = 1 \Rightarrow \int_c \mathbf{F} \cdot d\mathbf{r} = \int_{c_1} \mathbf{F} \cdot d\mathbf{r}_1 + \int_{c_2} \mathbf{F} \cdot d\mathbf{r}_2 = 2$

(b) Since$f(x,y) = x^3y^2$is potential function for $\mathbf{F}$, $\int_{(-1,1)}^{(1,1)} \mathbf{F} \cdot dr = f(1,1) - f(-1,1) = 2$

Exercise 18.

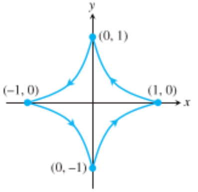
Evaluate the line integral $\int_C 2x\cos y \; dx - x^2 \sin y \; dy$ *along the following paths* C *in the xy-plane.*

(a) *The parabola* $y = (x-1)^2$ *from* $(1,0)$ *to* $(0,1)$

(b) *The line segment from* $(-1,\pi)$ *to* $(1,0)$

(c) *The x-axis from* $(-1,0)$ *to* $(1,0)$

(d) *The asteroid* $r\,(t) = \left(\cos^3 t\right) + \left(\sin^3 t\right) j, 0 \leq t \leq 2\pi,$ *counterclockwise form* $(1,0)$ *back to* $(1,0)$



Solution for Exercise 18

$\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -2x \ \sin \ y = \frac{\partial M}{\partial y} \Rightarrow$ **F**is conservative $\Rightarrow$ there exists an f so that $\mathbf{F} = \nabla f$; $\frac{\partial f}{\partial x} = 2x \ \cos \ y \Rightarrow f(x,y,z) = x^2 \cos \ y + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = -x^2 \sin \ y + \frac{\partial g}{\partial y} = -x^2 \sin y \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z) \Rightarrow f(x,y,z) = x^2 \cos \ y + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x,y,z) = x^2 \cos \ y + C \Rightarrow \mathbf{F} = \nabla(x^2 \cos \ y)$

(a) $\int_c 2x \ \cos \ y \ dx - x^2 \sin \ y \ dy = [x^2 \cos \ y]_{(1,0)}^{(0,1)} = 0 - 1 = -1$

(b) $\int_c 2x \ \cos \ y \ dx - x^2 \sin \ y \ dy = [x^2 \cos \ y]_{(-1,\pi)}^{(1,0)} = 1 - (-1) = 2$

(c) $\int_c 2x \ \cos \ y \ dx - x^2 \sin \ y \ dy = [x^2 \cos \ y]_{(-1,0)}^{(1,0)} = 1 - 1 = 0$

(d) $\int_c 2x \ \cos \ y \ dx - x^2 \sin \ y \ dy = [x^2 \cos \ y]_{(1,0)}^{(1,0)} = 1 - 1 = -0$

Exercise 19.

1. (a) Exact differential form : *How are the constants a, b, and c related if the following differential form is exact?*

$$\left(ay^2 + 2czx\right) dx + y\left(bx + cz\right) dy + \left(ay^2 + cx^2\right) dz$$

   (b) Gradient field : *For what values of b and c will*

$$F = \left(y^2 + 2czx\right) i + y\left(bx + cz\right) j + \left(y^2 + cx^2\right) k$$

   be a gradient field?

2. Gradient of a line integral : *Suppose that $F = \nabla f$ is a conservative vector field and*

$$g\left(x, y, z\right) = \int_{(0,0,0)}^{(x,y,z)} F \cdot dr.$$

   Show that $\nabla g = F$.

Solution for Exercise 19

1. (a) If the differential from is exact, then $\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} \Rightarrow 2ay = cy$ for all $y \Rightarrow 2a = c$, $\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \Rightarrow 2cx = 2cx$ for all x, and $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \Rightarrow$ by $= 2ay$ for all $y \Rightarrow b = 2a$ and $c = 2a$

   (b) $\mathbf{F} = \nabla f \Rightarrow$ the differential form with $a = 1$ in part (a) is exact $\Rightarrow b = 2$ and $C = 2$

2. $\mathbf{F} = \nabla f \Rightarrow g(x, y, z) = \int_{(0,0,0)}^{(x,y,z)} \mathbf{F} \cdot d\mathbf{r} = \int_{(0,0,0)}^{(x,y,z)} \nabla f \cdot d\mathbf{r} = f(x, y, z) - f(0, 0, 0) \Rightarrow \frac{\partial g}{\partial x} = \frac{\partial f}{\partial x} - 0$, $\frac{\partial g}{\partial y} = \frac{\partial f}{\partial y} - 0$ and $\frac{\partial g}{\partial z} = \frac{\partial f}{\partial z} = 0 \Rightarrow \nabla g = \nabla f = \mathbf{F}$, as claimed

Exercise 20.

1. Path of least work : *You have been asked to find the path along which a force field F will perform the least work in moving a particle between two locations. A quick calculation on your part shows* F *to be conservative. How should you respond? Give reasons for your answer.*
2. A revealing experiment : *By experiment, you find that a fore field* F *performs only half as much work in moving an object along path C_1 from A to B as it does in moving the object along path C_2 form A to B. What can you conclude about F? Give reasons for your answer.*
3. Work by a constant force : *Show that the work done by a constant force field $F = ai + bj + ck$ in moving a particle along any path from A to B is $W = F \cdot \bar{AB}$.*

Solution for Exercise 20

1. The path will not matter; the work along any path will be the same because the field is conservative.
2. The field is not conservative, for otherwise the work would be the same along C_1 and C_2.
3. Let the coordinates of points A and B be$(x_A, y_A, z_A) and (x_B, y_B, z_B)$, respectively. The forve $\mathbf{F} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is conservative because all the partial derivatives of M,N and P are zero. Therefore, the potential function is $f(x, y, z) = ax + by + cz + C$, and the work done by the force in moving a particle along any side pathfrom A to B is
$f(B) - f(A) = f(x_B, y_B, z_B) - f(x_A, y_A, z_A) =$
$(ax_B + by_B + c_B + C) - (ax_A + by_A + cz_A + C) = a(x_B - x_A) + b(y_B - y_Z) + c(z_B - z_A) = \mathbf{F} \cdot \overrightarrow{BA}$

Exercise 21.

(a) *Find a potential function for the gravitational field*

$$F = -GmM \frac{xi + yj + zk}{(x^2 + y^2 + z^2)^{3/2}}$$

(G, m, and M are constants).

(b) *Let P_1 and P_2 be points at distance s_1 and s_2 from the origin. Show that the work done by the gravitational field in part (a) in moving a particle form P_1 to P_2 is*

$$GmM \left(\frac{1}{s_2} - \frac{1}{s_1} \right)$$

Solution for Exercise 21

(a) Let $-GmM = C \Rightarrow \mathbf{F} = C\left[\frac{x}{(x^2+y^2+z^2)^{3/2}}\mathbf{i} + \frac{y}{(x^2+y^2+z^2)^{3/2}}\mathbf{j} + \frac{z}{(x^2+y^2+z^2)^{3/2}}\mathbf{k}\right] \Rightarrow \frac{\partial P}{\partial y} = \frac{-3yzc}{(x^2+y^2+c^2)^{5/2}} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = \frac{-3xzC}{(x^2+y^2+c^2)^{5/2}} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \frac{-3xyC}{(x^2+y^2+z^2)^{5/2}} = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F} = \nabla f$ for some f; $\frac{\partial f}{\partial x} = \frac{xC}{(x^2+y^2+z^2)^{3/2}} \Rightarrow f(x,y,z) = -\frac{C}{(x^2+y^2+z^2)^{1/2}} + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{yC}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial g}{\partial y} = \frac{yC}{(x^2+y^2+z^2)^{3/2}} \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z) \Rightarrow \frac{\partial f}{\partial z} = \frac{zC}{(x^2+y^2+z^2)^{3/2}} + h'(x) = \frac{zC}{(x^2+y^2+z^2)^{3/2}} \Rightarrow h(z) = C_1 \Rightarrow f(x,y,z) = -\frac{C}{(x^2+y^2+z^2)^{1/2}} + c_1.\text{Let} C_1 = 0 \Rightarrow f(x,y,x) = \frac{GmM}{(x^2+y^2+z^2)^{1/2}}$ is potential function for **F**.

(b) If s is the distance of (x,y,z) from the origin, then $s = \sqrt{x^2+y^2+z^2}$. The work done by the gravitational field

$\mathbf{F}$ *iswork* $= \int_{p_1}^{p_2} \cdot d\mathbf{r} = \left[\frac{GmM}{\sqrt{x^2+y^2+z^2}}\right]_{P_1}^{P_2} = \frac{GmM}{s2} - \frac{GmM}{s_2} - \frac{GmM}{x_1} = GmM\left(\frac{1}{x_2} - \frac{1}{s_1}\right)$, as claimed.

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